Available online at www.sciencedirect.com
SCIENCE @ DIRECT•
Applied Mathematics Letters 17 (2004) 509–517
Applied Mathematics Letters
www.elsevier.com/locate/aml

A Variant of Cauchy's Method with Accelerated Fifth-Order Convergence

M. GRAU AND M. NOGUERA
Department of Applied Mathematics II, Technical University of Catalonia
UPC, CN, Ed. Omega, 209
Jordi Girona 1-3, 08034 Barcelona, Spain
<grau><noguera>@ma2.upc.es

(Received August 2002; revised and accepted August 2003)

Abstract—We suggest an improvement to the iteration of Cauchy's method viewed as a generalization of possible improvements to Newton's method. Two equivalent derivations of Cauchy's method are presented involving similar techniques to ones that have been proved successfully for Newton's method. First, an adaptation of an auxiliary function that gives the new iteration function, and secondly, a symbolic computation that allows us to find the best coefficients with regard to the local order of convergence. The theoretical and computational order of convergence, for all functions tested, was five or more. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Cauchy's formula, Nonlinear equations, Iterative methods, Order of convergence.

1. INTRODUCTION

The Newton's method that approximates the root of a nonlinear equation in one variable using the value of the function and its derivative is probably the best known and most widely used algorithm, and if it is a simple root and the second derivative of the function is bounded in a neighborhood of this root, it converges to the root quadratically. There are improvements to the formula at the expense of an additional evaluation of the first derivative, additional evaluation of the function or a change in the point of evaluation [1]. Then the order of convergence is increased to three in the neighborhood of the root.

In this paper, we present a new method that is an improvement to the iteration function of Cauchy's method [1,2]. It uses the additional value of the second derivative and is a generalization of Newton's method with an order of convergence equal to three. The variant of Cauchy's method that we present is a new method that adds to the old one the evaluation of the function at another point (that iterated by Cauchy's method), and the order of convergence of the method increases by three to five.

Moreover, the current need to compute more digits more quickly and with more precision has led us to compute in FORTRAN with an arithmetic that uses a floating point representation with 108 decimal digits of mantissa.

 Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi:10.1016/j.aml.2003.08.005

2. NOTATION AND BASIC RESULTS

Let f be a function with a real root α. That is, the equation $f(x) = 0$ has a solution $f(\alpha) = 0$. We consider

$$f(t) = \sum_{j=0}^{k-1} \frac{f^{(j)}(x_i)}{j!}(t - x_i)^j + \frac{f^{(k)}(\xi_i(t))}{k!}(t - x_i)^k = P_{k-1}(t) + O\left[(t - x_i)^k\right], \tag{1}$$

where $\xi_i(t)$ lies in the interval determined by x_i, and P_{k-1} is the Taylor polynomial of degree $k-1$ whose first k derivatives agree with f at the point x_i, i.e., $P_k^{(\ell)}(x_i) = f^{(\ell)}(x_i)$, $\ell = 0, \dots, k-1$. Let the next approximation x_{i+1} be defined as the root of $P_k(x) = 0$ closest to x_i. Let the function that maps x_i into x_{i+1} be labelled g_k (more details can be found in [1,3,4]). Thus, $x_{i+1} = g_k(x_i)$. If we define

$$A_k(x) = \frac{f^{(k)}(x)}{k! f'(x)}, \tag{2}$$

it is known that g_k is a one-point iteration function of order k and asymptotic error constant $(-1)^k A_k(\alpha)$, since if $e_i = x_i - \alpha$, when i tends to infinite, x_i and x_{i+1} tend to α, and then $e_{i+1}/e_i^k \longrightarrow (-1)^k A_k(\alpha)$.

The iteration function, for $k = 2$, g_2 is that of the Newton method, which is simply $P_1(x_{i+1}) = 0 \Longrightarrow g_2 = x - u$, $u = f/f'$. For $k = 3$, the iteration function g_3 was studied by Cauchy (see [2,5]). We must solve $P_2(x_{i+1}) = 0$, and then $f(x) + f'(x)(t - x) + (1/2)f''(x)(t - x)^2 = 0$. The solution of this quadratic equation is

$$g_3 = x - \frac{f'}{f''} \pm \frac{\sqrt{(f')^2 - 2ff''}}{f''} = x - \frac{2u}{1 + (1 - 4A_2 u)^{1/2}}.$$

The order of the methods with iteration functions g_2 and g_3 are two and three, respectively, and their asymptotic error constants are $A_2(\alpha)$ and $A_3(\alpha)$, respectively.

There are several methods for computing roots of nonlinear functions originated from Newton's method. They define a multipoint iteration function such that the order of convergence is improved.

The first method that we present has local order three and it is obtained from the classical Newton iteration function

$$g_2(x) = x - \frac{f(x)}{f'(x)} = x - u(x),$$

by substituting the function $f(x)$ by $f(x) + f(\tilde{x})$, where $\tilde{x} = g_2(x)$. As a result, we get

$$G_3(x) = x - \frac{f(x) + f(\tilde{x})}{f'(x)},$$

which gives an iteration function with two evaluations of the function and one of its derivative.

An alternative construction of this method comes from the function (see [6])

$$H_3(t) = f(x) + f(t) + (t - x)f'(x). \tag{3}$$

If we assume $H_3(g) = 0$, then $f(x) + f(g) + (g - x)f'(x) = 0$, and we get

$$g = x - \frac{f(x) + f(g)}{f'(x)},$$

where we take $g = g_2$ on the right-hand side (also see [1]).

Another way to introduce the multipoint iteration function is to consider the function

$$K(t) = f(x) + (t-x)\sum_{i=1}^{r} \alpha_i f'\left[x + \beta_i(t-x)\right], \tag{4}$$

and we impose the conditions $K^{(\ell)}(x) = f^{(\ell)}(x)$, $\ell = 1, \ldots, r$. Then,

$$\ell \sum_{i=1}^{r} \alpha_i \beta_i^{\ell-1} = 1, \qquad \ell = 1, \ldots, r, \tag{5}$$

is a linear system of r equations in $2r$ unknowns with a Vandermonde determinant which depends on β_i. Hence, the r parameters β_i may be chosen arbitrarily as long as the r values chosen are distinct. System (5) will then determine the α_i uniquely. If we specialize (4) to the case $r = 2$, with $\beta_1 = 0$, we find that

$$K(t) = f(x) + (t-x)\left\{\alpha_1 f'(x) + \alpha_2 f'\left(x + \beta_2(t-x)\right)\right\}$$

and the system to solve is

$$\begin{aligned} \alpha_1 + \alpha_2 &= 1, \\ 2\beta_2\alpha_2 &= 1. \end{aligned}$$

1. If $\beta_2 = 1$, then $\alpha_1 = \alpha_2 = 1/2$, and $K(t) = f(x) + (1/2)(t-x)[f'(x) + f'(t)]$. If we define g by the condition $K(g) = 0$, then $g = x - (2f(x)/(f'(x) + f'(g)))$, and if we set $g = g_2 = x - u$ on the right-hand side of the preceding expression, the result is the third-order iteration function

$$\bar{G}_3 = x - \frac{2f(x)}{f'(x) + f'(x - u(x))}, \tag{6}$$

which is a known result [6,7]. The derivative that appears in Newton's method is replaced by the average of derivatives evaluated at x and at the next Newton iterated point of x.
2. Another method of order three can be introduced if we take $\beta_2 = 1/2$, then $\alpha_1 = 1$, $\alpha_2 = 0$, and $K(t) = f(x) + (t-x)f'\{x + (1/2)(t-x)\}$. The iteration method yields

$$\tilde{G}_3 = x - \frac{f(x)}{f'\left(x - (1/2)u(x)\right)}. \tag{7}$$

This method was presented by Traub, in 1964 [1].

In what follows, we will describe a constructive way to get the iteration method G_3. It consists in defining the iteration function

$$\Phi(x) = x - \frac{1}{f'(x)}\left(\gamma_1 f(x) + \gamma_2 f(x - u(x))\right), \tag{8}$$

which is Newton's method for $\gamma_1 = 1$ and $\gamma_2 = 0$. Now, the following question arises. Is there a set of values of γ_i, $i = 1, 2$ such that the iteration method (8) has a maximum order of convergence? Is the G_3 method maximal?

An answer to the preceding questions can be stated as follows.

THEOREM 1. *Let denote $f : I \to \mathbf{R}$ where I is a neighborhood of α, a simple root of $f(x)$ ($f'(\alpha) \neq 0$). Assume that f has first and second derivatives in I. Then the iteration function* *defined by (8) has a maximum local order of convergence for $\gamma_1 = \gamma_2 = 1$ and satisfies the following error equation: $e_{n+1} = 2A_2^2 e_n^3 + O(e_n^4)$, where $e_n = x_n - \alpha$ and $A_2 = f^{(2)}(\alpha)/2!f'(\alpha)$.*

PROOF. Let denote e by $e_n = x_n - \alpha$, and A_k by $A_k(\alpha)$, defined in (2); we use the following Taylor expansions:

$$f(x_n) = f'(\alpha)\left(e + A_2e^2 + A_3e^3 + O\left(e^4\right)\right) \tag{9}$$

and

$$f'(x_n) = f'(\alpha)\left(1 + 2A_2e + 3A_3e^2 + O\left(e^3\right)\right). \tag{10}$$

Dividing (9) by (10), $f(x_n)/f'(x_n) = e - A_2e^2 + 2(A_2^2 - A_3)e^3 + O(e^4)$, and then

$$\tilde{x}_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + A_2e^2 + 2\left(A_3 - A_2^2\right)e^3 + O\left(e^4\right). \tag{11}$$

Again by (11) and the Taylor expansions,

$$f\left(\tilde{x}_n\right) = f'(\alpha)\left(A_2e^2 + 2\left(A_3 - A_2^2\right)e^3\right) + O\left(e^4\right). \tag{12}$$

Adding (9), multiplied by γ_1, and (12), multiplied by γ_2, and dividing by (10),

$$\frac{\gamma_1 f(x_n) + \gamma_2 f\left(\tilde{x}_n\right)}{f'\left(x_n\right)} = \gamma_1 e + \left(\gamma_2 - \gamma_1\right)A_2e^2 + 2\left(\left(\gamma_2 - \gamma_1\right)A_3 + \left(\gamma_1 - 2\gamma_2\right)A_2^2\right)e^3 + O\left(e^4\right). \tag{13}$$

Thus, from $x_{n+1} = x_n - (\gamma_1 f(x_n) + \gamma_2 f(\tilde{x}_n))/f'(x_n)$, we get

$$e_{n+1} = \left(1 - \gamma_1\right)e_n + \left(\gamma_1 - \gamma_2\right)A_2e_n^2 + 2\left(\left(\gamma_1 - \gamma_2\right)A_3 + \left(2\gamma_2 - \gamma_1\right)A_2^2\right)e_n^3 + O\left(e_n^4\right). \tag{14}$$

Equation (14) establishes the maximum order of convergence for $\gamma_1 = 1$ and $\gamma_2 = 1$; since $e_{n+1} = 2A_2^2e_n^3 + O(e_n^4)$, the local order is three and there is no other iteration method from (8), varying the values of the coefficients γ_1 and γ_2, with order greater than or equal to three. ∎

2.1. The Main Result

In what follows, we apply the procedure used for building up the iteration function G_3 as an improvement of Newton's method, to a new iteration function G_5 that improves the order of Cauchy's method by two.

One possible way of building up this method comes from the auxiliary function

$$H_5(t) = f(x) + f(t) + (t - x)f'(x) + \frac{(t-x)^2}{2}f''(x), \tag{15}$$

where $H_5(t) = H_3(t) + ((t - x)^2/2)f''(x)$, and $H_3(t)$ is defined as in (3).

If we define g by the condition $H_5(g) = 0$, then g is a root of the quadratic equation $f(x) + f(g) + f'(x)(g - x) + (1/2)f''(x)(g - x)^2 = 0$, and the solution is

$$g - x = -\frac{2\left[f(x) + f(g)\right]/f'(x)}{1 + \left[1 - 2\left[f(x) + f(g)\right]f''(x)/f'(x)^2\right]^{1/2}}, \tag{16}$$

which consists of the iteration method of order five, G_5, if we set $g = g_3$, that is, Cauchy's method, into the right-hand side of the preceding expression. Then the variant and improved iteration method is defined, from (16), by

$$G_5\left(x_n\right) = x_n - \frac{2\left(u_n + \tilde{u}_n\right)}{1 + \left[1 - 4\left(u_n + \tilde{u}_n\right)\left(A_2\right)_n\right]^{1/2}}, \tag{17}$$

where $\tilde{u}_n = f(\tilde{x}_n)/f'(x_n)$, $\tilde{x}_n = g_3(x_n)$, and $(A_2)_n = A_2(x_n)$.

Next, we can construct G_5, defined by equation (17), in the same way as we did with G_3, that is, as a linear combination of evaluations of the function f at the points x and $g_2(x) = x - u(x)$. In this case, the generalization of Cauchy's method requires the evaluation of the function f at the points x and $g_3(x)$ with $g_3 = x - 2u/(1 + (1 - 4uA_2)^{1/2})$. Then, the iteration function, as in (8), is defined by

$$\Psi(x) = x - \frac{2\left[\gamma_1 f(x) + \gamma_2 f\left(g_3(x)\right)\right]/f'(x)}{1 + \left[1 - 2\left[\gamma_1 f(x) + \gamma_2 f\left(g_3(x)\right)\right] f''(x)/f'(x)^2\right]^{1/2}}, \tag{18}$$

that is, Cauchy's method for $\gamma_1 = 1$ and $\gamma_2 = 0$. Here, the following question arises: Is there a set of values of γ_i, $i = 1, 2$ such that the iteration method (18) has a maximum order of convergence? Is the G_5 method maximal?

An answer to the preceding questions can be stated as follows.

THEOREM 2. *Let denote $f : I \to \mathbf{R}$ where I is a neighborhood of α that is a simple root of $f(x)$, ($f'(\alpha) \neq 0$). Assume that f has first, second, and third derivatives in I. Then the new method defined by (18) has a maximum order of convergence equal to five for $\gamma_1 = \gamma_2 = 1$ and satisfies the following error equation: $e_{n+1} = 3A_3^2 e_n^5 + O(e_n^6)$, where $e_n = x_n - \alpha$ and $A_3 = f^{(3)}(\alpha)/3!f'(\alpha)$.*

PROOF. If we use the Taylor expansions as in (9)–(13), after easy albeit cumbersome manipulations, we obtain

$$\begin{aligned} e_{n+1} - \alpha = e_n - \alpha - \{&\gamma_1 e_n + A_2\gamma_1(\gamma_1 - 1)e_n^2 + \left[-A_3\left(2\gamma_1 + \gamma_2 - 3\gamma_1^2\right) + 2A_2^2\left(\gamma_1 - 2\gamma_1^2 - \gamma_1^3\right)\right]e_n^3 \\ &+ \left[A_2^3\left(-4\gamma_1 + 13\gamma_1^2 - 14\gamma_1^3 + 5\gamma_1^4\right) + A_2A_3\left(7\gamma_1 + 2\gamma_2 - 19\gamma_1^2 - 2\gamma_1\gamma_2 + 12\gamma_1^3\right)\right. \\ &\left.+ A_4\left(-3\gamma_1 - 3\gamma_2 + 6\gamma_1^2\right)\right]e_n^4 + \left[A_2^4\left(8\gamma_1 - 38\gamma_1^2 + 66\gamma_1^3 - 50\gamma_1^4 + 14\gamma_1^5\right)\right. \\ &+ A_2^2A_3\left(-20\gamma_1 - 4\gamma_2 + 10\gamma_1\gamma_2 + 83\gamma_1^2 - 108\gamma_1^3 - 6\gamma_1^2\gamma_2 + 45\gamma_1^4\right) \\ &+ A_2A_4\left(10\gamma_1 + 6\gamma_2 - 34\gamma_1^2 - 6\gamma_1\gamma_2 + 24\gamma_1^3\right) + A_3^2\left(6\gamma_1 - 21\gamma_1^2 - 6\gamma_1\gamma_2 + 18\gamma_1^3\right) \\ &\left.+ A_5\left(-4\gamma_1 - 6\gamma_2 + 10\gamma_1^2\right)\right]e_n^5\} + O\left(e_n^6\right). \end{aligned}$$

This last equation establishes the maximum order of convergence for $\gamma_1 = \gamma_2 = 1$; since $e_{n+1} = 3A_3^2 e_n^5 + O(e_n^6)$, the local order is five and there is no other iteration method from (18), varying the values of the coefficients γ_1 and γ_2, with order greater than or equal to five. Note that only one additional evaluation of the function improves the order of convergence by two unities. ∎

We can summarize as follows. The iteration function defined by

$$\Psi(x) = x - \frac{2\left[f(x) + f\left(g_3(x)\right)\right]/f'(x)}{1 + \left[1 - 2\left[f(x) + f\left(g_3(x)\right)\right] f''(x)/f'(x)^2\right]^{1/2}},$$

where g_3 is the classical Cauchy method has $O(e_n^5)$ local order of convergence.

Table 1. Test functions, their roots and their initial points (IP).

Function	Root	IP
$f_1(x) = x^3 + 4x^2 - 10$	1.365 ...	2.5
$f_2(x) = \sin^2(x) - x^2 + 1$	1.404 ...	3.5
$f_3(x) = x^2 - \exp(x) - 3x + 2$	0.257 ...	−5.0
$f_4(x) = \cos(x) - x$	0.739 ...	3.0
$f_5(x) = (x - 1)^3 - 1$	2.0	2.5
$f_6(x) = x^3 - 10$	2.154 ...	1.0
$f_7(x) = x\exp\left(x^2\right) - \sin^2(x) + 3\cos(x) + 5$	−1.207 ...	−0.5
$f_8(x) = x^2\sin^2(x) + \exp\left(x^2\cos(x)\sin(x)\right) - 28$	4.622 ...	4.7
$f_9(x) = \exp\left(x^2 + 7x - 30\right) - 1$	3.0	2.9

3. EXAMPLES

We have tested these methods with nine functions and a FORTRAN multiprecision arithmetic floating point. We computed the root of each function, and the order of convergence for each method.

3.1. Technical Details

We used Bailey's routines for the FORTRAN multiprecision arithmetic floating point, based on the fast Fourier transform (FFT) [8]. In all computations, we used 108 decimal digits (or 16 words) for the representation of the floating point. The method is stopped when $|x_n - \alpha| < 10^{-100}$, where α is the exact root computed with 120 digits.

The order of convergence is defined by ρ such that $\lim_{n\to\infty}(e_{n+1}/e_n^\rho) = c \neq 0$. Then the order of convergence can be approximated by $\rho \approx \log|(x_{n+1}-\alpha)/(x_n-\alpha)|/\log|(x_n-\alpha)/(x_{n-1}-\alpha)|$. Table 2 shows the results of the computed order for each of the methods defined above.

Table 2. Number of evaluations of the function and their derivatives for each method, and the order and the efficiency of the iterative functions.

	f	f'	f''	ρ	E_1	E_2
g_2	1	1	0	2	1.414	1.440
$\bar{G}_3$	1	2	0	3	1.442	1.479
$\tilde{G}_3$	1	2	0	3	1.442	1.479
G_3	2	1	0	3	1.442	1.514
g_3	1	1	1	3	1.442	1.565
G_5	2	1	1	5	1.495	1.615

3.2. Test Functions

The test functions are the same as in Weerakoon and Fernando [6]. Table 1 shows the expression, the root that we have calculated and the initial point of iteration. All roots are similar in magnitude and the initial point is the same for all methods.

In order to show the difficulty in calculating the root of these functions, we have plotted them around their root, although the axes are x in the abscissas, versus $\operatorname{arcsinh}(f_i(x))$ in the ordinates. The latter axis is similar to the logarithmic axis, except that we must plot zero and negatives values.

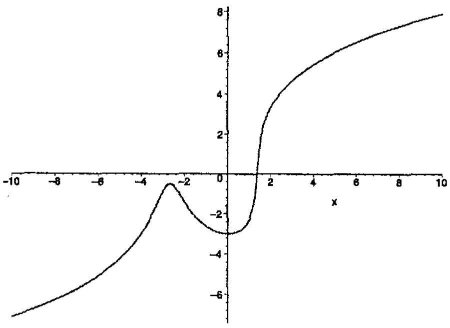
It should be noted that in the function $f_8(x)$ the initial point is very critical, because the root is not unique, and the method can converge to another root.

If we wish to calculate the cost of each method, we need the number of evaluations of the function and their derivatives at one iteration for each method. Table 2 shows this concept and the computational efficiency, (see [1]): $E = \rho^{1/\theta}$, with $\theta = \sum \theta_j$, where θ_j is the cost of evaluating $f^{(j)}$. We have taken the cost of $f^{(j)}$ as being equal to 1 in column E_1, and an average value of the real cost of $f_i^{(j)}$ $i = 1, \dots, 9$ in column E_2.

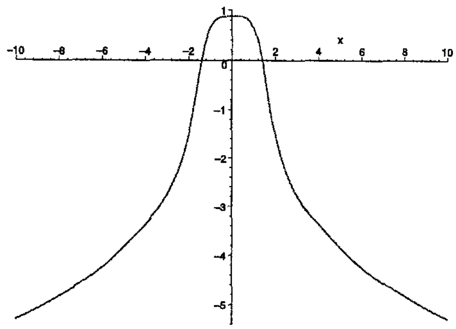
3.3. Numerical Results

Table 3 shows the number of iterations necessary to reach the root with the described precision for each function and the methods g_2 (the classical Newton method), $\bar{G}_3$ (Hummel and Seebeck's variant), $\tilde{G}_3$ (Traub's variant), G_3 (Traub's variant and maximal method), g_3 (the classical Cauchy method), and G_5 (the variant of the Cauchy method).

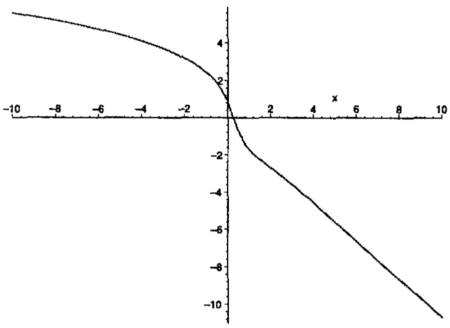
The number of iterations marked with an asterisk are those for which convergence was very slow at the first iterations, with a very low order of convergence far removed from their order. Finally, number of iterations given with two values indicate that the method converges only with



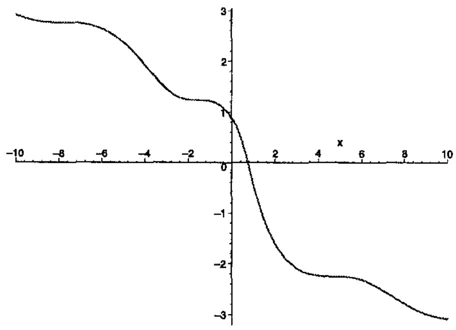
$f_1(x) = x^3 + 4x^2 - 10$



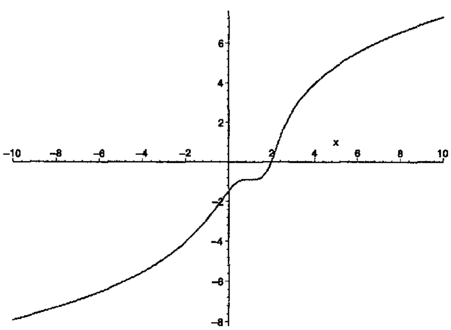
$f_2(x) = \sin^2(x) - x^2 + 1$



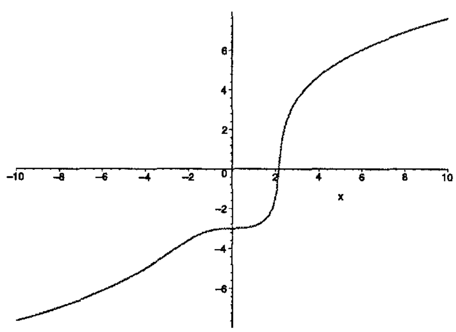
$f_3(x) = x^2 - \exp(x) - 3x + 2$



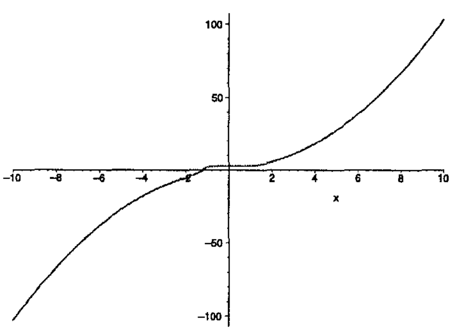
$f_4(x) = \cos(x) - x$



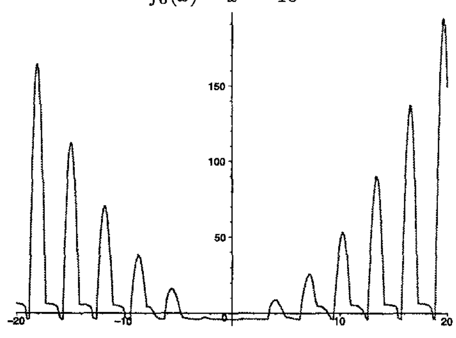
$f_5(x) = (x-1)^3 - 1$



$f_6(x) = x^3 - 10$

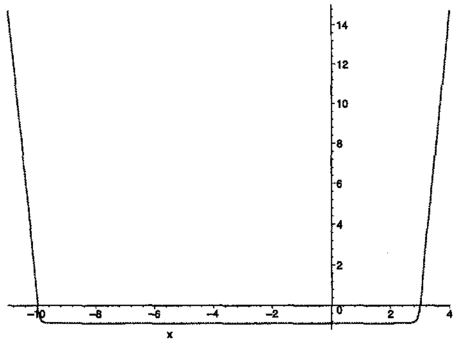


$f_7(x) = x\exp(x^2) - \sin^2(x)$
$+3\cos(x) + 5$



$f_8(x) = x^2\sin^2(x)$
$+\exp(x^2\cos(x)\sin(x)) - 28$

Figure 1. Plots of test function, x versus $\operatorname{arcsinh}(f_i(x))$.



$f_9(x) = \exp(x^2 + 7x - 30) - 1$

Figure 1. (cont.).

the last iterations (the number in parentheses), whereas at the first iterations (the difference between the two numbers), the error does not decrease significantly.

From Tables 3 and 4, we can establish that the higher number of iterations of the functions $f_i(x)$, $i = 6, \dots, 9$, in the g_2 and $\bar{G}_3$ methods result in a greater computational time. We would emphasize the low cost of the iteration function G_5, with a better rate of computational time than the other methods considered, with only a few exceptions. In general, the results are excellent, the order is maximized, and the computational time is lower that we expected.

Table 3. Number of iterations for each method and each function (see Section 3.3 for details).

	g_2	$\bar{G}_3$	$\tilde{G}_3$	G_3	g_3	G_5
$f_1(x)$	8	6	5	6	5	4
$f_2(x)$	9	6	6	6	5	4
$f_3(x)$	9	6	6	6	6	4
$f_4(x)$	9	10(6)	5	5	6	4
$f_5(x)$	8	6	5	6	5	4
$f_6(x)$	10(*)	7	6	12(6)	5	4
$f_7(x)$	12(*)	13(*)	7	7	6	5
$f_8(x)$	12(*)	10(*)	7	7	6	5
$f_9(x)$	10(*)	7	6	10(*)	6	4

Table 4. The rate between the computational time for each method and each function in comparison with the Newton method (g_2).

	g_2	$\bar{G}_3$	$\tilde{G}_3$	G_3	g_3	G_5
$f_1(x)$	1	0.68	0.39	1.07	1.07	0.68
$f_2(x)$	1	0.73	0.77	0.75	0.77	0.50
$f_3(x)$	1	0.66	0.63	0.91	0.63	0.62
$f_4(x)$	1	1.65	0.65	0.70	0.65	0.28
$f_5(x)$	1	0.37	0.31	0.68	0.68	0.37
$f_6(x)$	1	0.88	1.09	2.83	1.88	0.88
$f_7(x)$	1	2.19	1.00	0.81	0.68	0.51
$f_8(x)$	1	1.71	0.87	1.45	0.42	0.45
$f_9(x)$	1	1.91	1.91	3.00	1.91	1.03

4. CONCLUDING REMARKS

An improvement in the iteration of Cauchy's method is presented from two equivalent derivations. One of them is a symbolic computation that allows us to find the best coefficients with regard to the local order of convergence.

Many numerical applications use high precision in their computations. In these types of applications, numerical methods of high order are important. The results of these numerical experiments show that the five-order method G_5, introduced in this work, associated with a multiprecision arithmetic floating point is very useful, because it yields a clear reduction in iterations. In almost all test functions, the computational time is lower than in the classical Newton method.

REFERENCES

1. J.F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, NJ, (1964).
2. D.M. Young and R.T. Gregory, *A Survey of Numerical Methods*, Dover, New York, (1988).
3. M. Grau and M. Noguera, *Cálculo Numérico*, Ediciones UPC, Barcelona, (2001).
4. J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Springer-Verlag, New York, (1983).
5. A.L. Cauchy, Sur la détermination approximative des racines d'une équation algebrique ou trascendente, *Oeuvres Complètes (II)* **4**, 573–609, (1882–1938).
6. S. Weerakoon and T.G. I. Fernando, A variant of Newton's method with accelerated third-order convergence, *Appl. Math. Lett.* **13** (8), 87–93, (2000).
7. P.M. Hummel and C.L. Seebeck, A generalization of Taylor's expansion, *Amer Math Monthly* **56**, 243–247, (1949).
8. D.H. Bailey, Multiprecision translation and execution of Fortran programs, *ACM Transactions on Mathematical Software* **19** (3), 288–319, (1993).